Journal of Statistical Physics, Vol. 46, Nos. 5/6, 1987

Spectral Fluctuations and Zeta Functions

N. L. Balazs,[1] C. Schmit,[2] and A. Voros[3]

La place n'est point de celles que l'on puisse emporter de haute lutte; il faut l'attaquer successivement sur toute sorte de points et se contenter d'avantages partiels [J. Hadamard, *Œuvres*, IV (CNRS, 1968), p. 1954]

Received November 19, 1986

We study theoretically and numerically the role of the fluctuations of eigenvalue spectra $\{\mu_n\}$ in a particular analytical continuation process applied to the (generalized) zeta function $Z(s) = \sum_n \mu_n^{-s}$ for s large and positive. A particularly interesting example is the spectrum of the Laplacian on a triangular domain which tessellates a compact surface of constant negative curvature (of genus two). We indeed find that the fluctuations restrict the abscissa of convergence, and also affect the rate of convergence. This then initiates a new approach to the exploration of spectral fluctuations through the convergence of analytical continuation processes.

KEY WORDS: Quantum chaos; spectral fluctuations; zeta function; trace identities; billiards.

1. INTRODUCTION

A self-adjoint operator is *completely* characterized by its eigenvalues and eigenfunctions. However, very often one limits one's attention to the collection of eigenvalues, to its spectrum. For example, in quantum physics the spectrum of the Hamiltonian operator gives the allowed energy values of

[1] Department of Physics, State University of New York at Stony Brook, Stony Brook, New York 11794; work done while at the Service de Physique Théorique, Saclay, and Institut de Physique Nucléaire, Orsay.

[2] Division de Physique Théorique (Laboratoire Associé au CNRS), Institut de Physique Nucléaire, 91406 Orsay, France; member of CNRS.

[3] Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France; member of CNRS.

the dynamical system described by this Hamiltonian. In acoustics the normal mode frequencies furnish the frequencies generated by a musical instrument, such as a pipe or a drum. The latter problem, in turn, leads to the general mathematical question of determining the spectrum of the Laplacian on closed surfaces with or without boundaries.[1] Associated with this is the inverse problem, the possible reconstruction of the vibrating surface through the knowledge of its spectrum, expressed once by Mark Kac[2] by the pregnant question, "Can one hear the shape of a drum?" To honor his memory we shall address ourselves briefly to a ramification of this question and study a possible characterization of the spectral fluctuations: "How noisy is your drum?" The motivation comes from quantum dynamics. Consider a classical Hamiltonian system exhibiting chaotic behavior in the phase space (being, say, Bernoullian).[3] Upon quantization, let the quantum Hamiltonian have a discrete spectrum. Can one tell, looking at this spectrum *alone*, that in the *classical limit* the motion of the system is chaotic? Conversely, can one deduce from the chaotic properties of the classical motion particular features of the quantal spectrum? This problem is largely unsolved and is under active investigation.[4] In particular, it has been conjectured[5] that the fluctuations of the spectrum around its mean distribution yield such a criterion. Actually, numerical experimentation indicates that these fluctuations are much diminished (in a well-defined sense) if the associated classical Hamiltonian is chaotic.

Our present purpose here is the discussion of a conjecture concerned with a convenient characterization of the fluctuations in the spectrum.[6]

2. SPECTRAL STORAGE

Consider the Hamiltonian $\hat{\mathscr{H}}$ and its spectrum of eigenvalues $\mu_0, \mu_1, \ldots, \mu_n, \ldots$ ordered in an increasing sequence. (If degeneracies are present, the eigenvalues are repeated according to their multiplicities. The zero of energy is so chosen that $\mu_0 > 0$.)

There are many different ways to store this set of numbers by associating functions with the set. Such functions are the generating function $g(z) = \sum_n \mu_n z^n$, the partition function $\Theta(t) = \sum_n e^{-t\mu_n}$, the trace of Green's function $R(\mu) = \sum_n [1/(\mu - \mu_n)]$, the Fredholm determinant $D(\mu) = \prod_n (1 - \mu/\mu_n)$, the zeta function $Z(s) = \sum_n \mu_n^{-s}$, etc. Of these, the generating function is little used, since it cannot conveniently be expressed as the trace of an operator function of $\hat{\mathscr{H}}$. The partition function is the trace of the heat operator $\exp(-t\hat{\mathscr{H}})$; the others can be thought of as different integral transforms of the heat operator; for example, the resolvent is its Laplace transform, while the zeta function is proportional to its Mellin transform. Different integral transforms stress different properties of the spectrum through the analytical dependence of the transform on its variable. For this reason several of these transforms must be investigated to obtain a more detailed comprehension of the spectrum. In this note we concentrate on one particular case, the zeta function, since we believe that the latter is particularly suitable to study the *fluctuation* properties of the spectrum through its analyticity properties. In fact, we surmise that the *process* of analytical continuation of $Z(s)$, based on the defining series $\sum_{n=1}^{\infty} \mu_n^{-s}$ (with s or its real part being large and positive) will crucially depend on the nature of the fluctuations of the set μ_n.

To explain our beliefs and motivation we illustrate on Riemann's zeta function $\zeta(s)$ how the process of analytical continuation can be influenced by the presence or absence of fluctuations [the latter being the case for $\zeta(s)$].

3. THE EXAMPLE OF RIEMANN'S ZETA FUNCTION

Consider the equidistant spectrum $\mu_n = n$, with $n = 1, 2,...$, for example, the spectrum of the harmonic oscillator with a shift $+1/2$. In this case

$$Z(s) = \sum_{n=1}^{\infty} n^{-s} \equiv \zeta(s) \tag{1}$$

where $\zeta(s)$ is Riemann's zeta function, which can also be written as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\, dt}{e^t - 1} \tag{2}$$

Both definitions of $\zeta(s)$ converge for $\operatorname{Re} s > 1$.[7] A central question in the theory of the zeta function is to find an effective analytical extension of either formula to $\operatorname{Re} s \leqslant 1$. Two methods are in use.

(a) *For the series representation* the well-known trick is to express the tail of the series (which diverges for $s \leqslant 1$) in a form where analytical continuation is obvious. We then use the Euler–Maclaurin summation formula[8] to write $\sum_N^{\infty} n^{-s}$ as $\int_N^{\infty} n^{-s}\, dn$, amended by correction terms involving the Bernoulli numbers [see Eq. (7)],

$$\sum_{n=N}^{\infty} n^{-s} = -\frac{N^{1-s}}{1-s} + \frac{1}{2} N^{-s} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} s(s+1) \cdots (s+2m-2)\, N^{-s-2m+1} \tag{3}$$

(this being understood as an asymptotic expansion for $N \to \infty$).

Hence, we also have for all M, and $\operatorname{Re} s > 1$,

$$\zeta(s) = \lim_{N \to \infty} \left[\sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2} N^{-s} + \sum_{m=1}^{M} \frac{B_{2m}}{(2m)!} s(s+1) \cdots (s+2m-2) N^{-s-2m+1} \right] \tag{4}$$

The advantage of this new expression is that the limit $N \to \infty$ continues to exist as long as $\operatorname{Re} s > 2M - 1$, and being analytic in s, it does give a numerical representation of the analytical continuation of $\zeta(s)$, which holds for *arbitrary negative* s provided enough terms are included, i.e., for $2M > -1 - s$. In fact Eq. (4) is one of the classical methods to compute *numerically* $\zeta(s)$ for a variety of purposes. We note, however, that as s is made more and more negative, $\zeta(s)$ is computed as the difference of two *increasingly larger* quantities; this property will prove most useful later. Also, Eq. (4) exhibits the well-known polar singularity $1/(s-1)$ at $s = 1$ hence can be used to evaluate the *finite part* of $\zeta(s)$ at $s = 1$, as

$$FP\zeta(1) = \lim_{s \to 1} \left[\zeta(s) - \frac{1}{s-1} \right] = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right) = \gamma \tag{5}$$

(this is Euler's constant). Equation (4) can also be *differentiated* to yield, for instance,

$$\zeta'(0) = \lim_{N \to \infty} \left[\sum_{n=1}^{N} (-\log n) + N(\log N - 1) + \frac{1}{2} \log N \right] = -\frac{1}{2} \log 2\pi \tag{6}$$

where we have used Stirling's formula for $\log N!$.

(b) *As for the integral representation of* $\zeta(s)$, one method for analytical continuation uses the asymptotic expansion of the integrand for $t \to 0^+$,

$$\frac{1}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^{n-1} \tag{7}$$

where B_n are the Bernoulli numbers (an expansion much used in evaluating integrals over the Planck distribution).

The expansion (7) can immediately be used for $\zeta(s)$. For the general $Z(s)$ function a more involved procedure is required, since the integral expression corresponding to (2) is not quite the same. Consequently, here, too, we proceed as in the general case. We write

$$\zeta(s) = \eta(s)/\Gamma(s) \tag{8}$$

with

$$\eta(s) = \int_0^\infty \Theta(t)\, t^{s-1}\, dt \tag{9}$$

In the present case $\Theta(t)$ is simply $(e^t - 1)^{-1}$, while in general $\Theta(t) = \sum_0^\infty e^{-t\mu_n}$. Both admit an asymptotic expansion as $t \to 0^+$. Equation (7) is exceptional in two respects: (i) it gives a convergent expansion, (ii) it proceeds in integral powers. In general[1]

$$\Theta(t) \simeq c_\alpha t^\alpha + c_\beta t^\beta + c_\gamma t^\gamma + \cdots \tag{10a}$$

with

$$\alpha < \beta < \gamma < \cdots \to +\infty, \qquad \alpha < 0 \tag{10b}$$

If $t \to +\infty$, $\Theta(t)$ is exponentially decreasing.

We now study the limit of convergence of $\eta(s)$ as s decreases. From Eq. (10), $\Theta(t)$ diverges as t^α for small t. Hence, from Eq. (9) we find that limit to be $\operatorname{Re} s = -\alpha$. However, we can extend the region of convergence, provided the series (10a) can be differentiated term by term, as is usually the case. Integrate by parts in (9),

$$\eta(s) = -\int_0^\infty [t^{-\alpha}\Theta(t)]' \frac{t^{s+\alpha}}{s+\alpha}\, dt \tag{11}$$

We now observe that the leading term c_α is suppressed by the differentiation; thus, the limit of convergence of this new integral becomes $\operatorname{Re} s = -\beta$, providing the analytical extension of $\eta(s)$ up to that point.

In addition we see from (11) that $\eta(s)$ has a pole at $s = -\alpha$, where the residue can be computed by reversing the partial integration,

$$\operatorname{Re} s_{s=-\alpha}\, \eta(s) = -\int_0^\infty [t^{-\alpha}\Theta(t)]'\, dt = t^{-\alpha}\Theta(t)\Big|_{t=0} = c_\alpha \tag{12}$$

Repeat now the same idea using

$$\eta(s) = +\int [t^{-\beta+\alpha+1}\theta_1(t)]' \frac{t^{s+\beta}}{(s+\beta)(s+\alpha)}\, dt \tag{13}$$

with $\theta_1(t) = [t^{-\alpha}\theta(t)]'$ [which behaves for small t as $c_\beta(\beta-\alpha)\, t^{\beta-\alpha-1}$]. The resulting integral converges now up to $\operatorname{Re} s = -\gamma$, providing an analytical extension up to this point and exhibiting a second pole at

$s = -\beta$, with residue c_β. Continuing this process, we construct $\eta(s)$ as a *meromorphic* function in the whole s plane with poles exclusively at $s = -\alpha$, $-\beta$, $-\gamma$, etc., having there the residues c_α, c_β, c_γ, etc.

Consequently, $Z(s) = \eta(s)/\Gamma(s)$ is also a meromorphic function in the whole plane, with analyticity properties determined by the interplay of the poles of $\eta(s)$ and $\Gamma(s)$. The latter has a pole at every negative integer and zero; this either will give a zero of $Z(s)$ or will cancel those poles of $\eta(s)$ that are associated with integral exponents in the expansion (10). A remaining pole generated by a nonintegral or negative exponent, say δ, has the residue $c_\delta/\Gamma(-\delta)$.

The cancellations at the negative integers $(-n)$ and at zero lead to the infinite sequence of *trace identities*[9]

$$Z(-n) \equiv (-1)^n c_n n!, \qquad n = 0, 1, 2,... \tag{14}$$

where by convention $c_n \equiv 0$ if n does not appear in the sequence $\alpha, \beta, \gamma,...$.

We now apply these results to Riemann's zeta function $\zeta(s)$. The only surviving pole of $\zeta(s)$ is at $s = 1$ with the residue $B_0 = 1$; on the other hand, the trace identities specify

$$\zeta(-n) = (-1)^n B_{n+1}/(n+1) \tag{15}$$

Since $B_k \equiv 0$ for odd $k > 1$, $\zeta(-2n) \equiv 0$ [except $\zeta(0) = -1/2$].

(c) Another way of continuing $\zeta(s)$ to the whole region $\operatorname{Re} s < 0$ *at once* is to use the functional equation for $\zeta(s)$. This method, however, is not available for general zeta functions, since it is the consequence of the *arithmetical* properties of $\zeta(s)$. Like the distribution of the complex zeros of $\zeta(s)$, the object of Riemann's hypothesis, etc., it reflects the regularity and exceptional character of an *equidistant* spectrum. Arithmetical properties are thus not expected to persist for a more general spectrum; therefore, it is unlikely that this alternative method of analytical continuation will have a counterpart for a generic spectrum.

4. THE GENERAL ZETA FUNCTION Z(*s*)

After this preparation, we consider the general situation with $\Theta(t) = \sum e^{-t\mu_n}$ and

$$Z(s) = \sum_{n=0}^{\infty} \mu_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \Theta(t)\, t^{s-1}\, dt \tag{16}$$

provided s or its real part is positive and large enough. The last equality is the analogue of Eqs. (8) and (9), as can be verified using the definition of $\Theta(t)$ and integrating term by term.

Continuing with the analogy, we now attempt in two different ways the analytical continuation of $Z(s)$ to smaller, and eventually negative s (or $\operatorname{Re} s$) values.

The continuation using the integral representation in (16) was already treated in Section 3 and can be used here provided that an asymptotic expansion like (10) exists for $\Theta(t)$ for small t; one of the results of that procedure has been the set of trace identities (14), $Z(-n) = (-1)^n c_n n!$.

The alternative method for analytical continuation starts with the series representation $Z(s) = \sum \mu_n^{-s}$ and its transcription using the Euler–Maclaurin summation formula analogously to Eqs. (3) and (4) for $\zeta(s)$. Here, too, we try to rewrite the tail of the series in a form where analytical continuation is transparent. *Formally*, the general Euler–Maclaurin summation formula gives[8]

$$\sum_{n=N}^{\infty} \mu_n^{-s} = \int_N^{\infty} \mu(n)^{-s}\, dn + \frac{1}{2} \mu_N^{-s} - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \left(\frac{d}{dn}\right)^{2m-1} \mu(n)^{-s} \bigg|_{n=N} \tag{17}$$

However, we now must observe that while the left-hand side is completely determined by the eigenvalue sequence μ_n, the right-hand side *requires the choice of a smooth function* $\mu(n)$ of the continuous index n, which replaces the discrete eigenvalue sequence μ_n in some suitable manner. In particular, the use of the Euler–Maclaurin summation formula (17) implies a smoothness, such that the derivatives $(d/dn)^{2m-1} \mu(n)^{-s}$ should decay to 0 as $n \to \infty$; this precludes oscillations of $\mu(n)$ for large values of n. This condition may then *prohibit* a function $\mu(n)$ that passes *exactly* through the eigenvalue sequence at all integer values of n, a prohibition giving rise to inevitable differences between $\mu(n)$ and μ_n. It is then natural to define the spectral fluctuations through the deviation of the exact sequence μ_n from a function $\mu(n)$ that is sufficiently smooth in the above sense. Since, in principle, many such smooth functions are conceivable, we must now make a choice. It seems, however, that only one function $\mu(n)$ is available in practice and even that is only asymptotically specified, through the use of the *Weyl formula* with corrections to all orders. This expression describes the asymptotic distribution of eigenvalues as

$$W(\mu) \sim \frac{c_\alpha}{\Gamma(1-\alpha)} \mu^{-\alpha} + \frac{c_\beta}{\Gamma(1-\beta)} \mu^{-\beta} + \frac{c_\gamma}{\Gamma(1-\gamma)} \mu^{-\gamma} + \cdots \tag{18}$$

This $W(\mu)$ approximates the spectral staircase function $N(\mu)$ (which counts the number of levels below μ) through a smoothing operation followed by an asymptotic expansion for large μ. See Ref. 6 and Appendix A, where the geometrical interpretation of the coefficients is also given. (Although there is no unique choice of the smoothing, the asymptoticity of the expansion obliterates this ambiguity.)

The Weyl function $W(\mu)$ naturally defines a continuous function $\mu(n)$ by solving the following equation for μ:

$$n+\tfrac{1}{2}=W(\mu) \tag{19}$$

We now motivate the appearance of the term 1/2 in this equation. Define a sequence $\mu'_0, \mu'_1, \mu'_2,\ldots$ by solving this equation for integral n, and consider the spectral staircase for this sequence μ'_n. By construction, $W(\mu)$ is the *best smooth approximation* to this staircase, intercepting each vertical step at its midpoint; altering 1/2 to any other value damages this property.

If the problem is one-dimensional, the term 1/2 also emerges inevitably from the Bohr–Sommerfeld quantization condition, since in this case $W(\mu)$ is $(1/2\pi)\oint p\,dq$ to lowest order. We must stress, however, that although Eq. (19) can be used as a quantization condition in *one* dimension (being equivalent to the Bohr–Sommerfeld rule), this is no longer true in higher dimensions. Thus, Eq. (19) *cannot be used* to predict individual eigenvalues μ_n, precisely because the deviations $(\mu_n-\mu'_n)$ will be in general quite large, as is indeed shown by numerical experiments. However, the same numerical experiments still show that the best average agreement between the exact staircase and the smooth curve given by Eq. (19) actually takes place through the use of the term 1/2.

We then insert into the Euler–Maclaurin formula, Eq. (17), the function $\mu(n)$ obtained from Eq. (19), resulting in

$$\begin{aligned}\sum_{n=N}^{\infty}\mu_n^{-s}=&\int_{\mu_N}^{\infty}\mu^{-s}\,dW(\mu)+\frac{1}{2}\mu_N^{-s}\\&-\sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}\left(\frac{d}{dW}\right)^{2m-1}\mu^{s}\bigg|_{\mu=\mu_N}\end{aligned} \tag{20}$$

Hence we obtain the *basic continuation formula* as

$$\begin{aligned}Z(s)\underset{N\to\infty}{\simeq}&\sum_{n=0}^{N-1}\mu_n^{-s}+\frac{1}{2}\mu_N^{-s}+\int_{\mu_N}^{\infty}\mu^{-s}\,dW(\mu)\\&-\sum_{m=1}^{M}\frac{B_{2m}}{(2m)!}\left(\frac{d}{dW}\right)^{2n-1}\mu_s\bigg|_{\mu=\mu_N}\end{aligned} \tag{21}$$

We now discuss the relationship implied here by the sign $\sim$.

Consider for a moment the one-dimensional harmonic oscillator. In this case Eq. (19) gives the exact eigenvalues, and thus, $\mu_n \equiv \mu'_n$; no fluctuations exist in the spectrum and one can proceed as with Riemann's zeta function. Then the right-hand side of (21) is an asymptotic expansion in powers of $1/N$. We truncate the series at M, far enough (depending on s) to make the rest vanish as $N \to \infty$. Then this limit is independent of M and gives the correct numerical value of the analytical continuation of $Z(s)$.

For a more general but still one-dimensional situation, the Bohr–Sommerfeld quantization condition states that $\mu_n \sim \mu'_n$ asymptotically to all orders for large n; thus, the fluctuations are still absent and the same results ensue.[10] Consequently, in the one-dimensional case the analytical continuation can be performed without further ado by taking the limit $N \to \infty$ in Eq. (21).

In more than one dimension, however, the situation can be quite different. Since now fluctuations in the spectral distribution will arise, neither the convergence of the right-hand side of (21) (as $N \to \infty$) nor its equality to be left-hand side can be taken for granted, except within the region $\operatorname{Re} s > -\alpha$. (Here the defining series $\sum \mu_n^{-s}$ converges anyway, and the Euler–Maclaurin correction terms are used only to accelerate the numerical convergence.)

We notice that if the convergence of Eq. (21) persists as s decreases beyond the abscissa of convergence of the series $\sum \mu_n^{-s}$, this basically arises through the cancellation of the partial sum $(\sum_0^{N-1} \mu_n^{-s} + \frac{1}{2}\mu_N^{-s})$ with $\int_{\mu_N}^{\infty} \mu^{-s}\, dW(\mu)$. Both quantities increase much faster than $Z(s)$ itself as s decreases, *making the cancellation process increasingly more sensitive to spectral fluctuations.*

Let the fluctuation be $\delta\mu_N$. Then the value of the partial sum is essentially changed by the variation of the last term, which is of the order of $-s\mu_N^{-s-1}\,\delta\mu_N$. The integral will change because of the variation in the lower limit, giving a change $\mu_N^{-s}\,\delta W(\mu_N)$. Since in two dimensions $W(\mu_N)$ grows linearly (and in higher dimensions even faster), we see that (a) the spectral fluctuations produce an effect in the integral term that is larger by at least a factor μ_N, (b) this effect, in turn, *grows exponentially* as s decreases. Thus, the nature of the putative convergence as a function of s will reflect the nature and size of the spectral fluctuations. In particular, the following options are present:

(a) The process cannot be made to converge at all.

(b) The process converges up to a natural abscissa of convergence s_0 and diverges for $\operatorname{Re} s \leqslant s_0$. In this situation the *location* of s_0 gives a natural measure of the fluctuations' size.

(c) The process converges for all s.

In addition, further refinements can aid us to extract more information about the fluctuations. Where convergence is actually achieved, *the rate of convergence* can still be used to measure the *extent* of the spectral fluctuations. If convergence is not achieved, two possibilities can occur. The Euler–Maclaurin method can be improved by some regularization (Cesaro summation, Gaussian smoothing, etc.), or an altogether different method must be invented [for example, the discovery of the functional equation for $\zeta(s)$].

The nature of the convergence for $s=0$ as $N \to \infty$ in Eq. (21) is particularly revealing. The trace identity gives the exact value c_0 for Z(0). Taking now the difference of the two sides of Eq. (21) at $s=0$, we obtain

$$c_0 - (N+1/2) - \left[\int_{\mu_N}^{\infty} \mu^{-s}\, dW(\mu) \right]_{s \to 0} \tag{22}$$

where on the left-hand side the term N comes from the sum, while the correction terms containing B_{2m} are omitted, since they tend to zero as $s \to 0$. We now show that c_0 minus the limit of the integral is in fact the Weyl function $W(\mu)$ of Eq. (18). Insert (18) in the integral for large s, evaluate the integral term by term; letting *now* s tend to zero, we find precisely $W(\mu_N) - c_0$; finally, the difference of the two sides of Eq. (21) at $s=0$ simplifies to

$$\mathrm{Z}(s) - \sum_{n=0}^{N-1} \mu_n^{-s} + \tfrac{1}{2}\mu_N^{-s} + \int_{\mu_N}^{\infty} \mu^{-s}\, dW(\mu) \bigg|_{s=0} = W(\mu_N) - (N + \tfrac{1}{2}) \tag{23}$$

Consequently, *the presence or absence of convergence at* $s=0$ *monitors the deviation of the actual distribution of eigenvalues from the one given by* $W(\mu)$.

In the next section we illuminate this discussion by numerical examples.

5. THE ROLE OF FLUCTUATIONS IN THE CONTINUATION PROCESS

In the previous sections we outlined the following idea. One defines Z(s) as a power series for real $s > -\alpha$ and uses the Euler–Maclaurin formula to attempt an analytical continuation for $s < -\alpha$. The surmise is that the success of this continuation process depends on the nature of the spectral fluctuations, and thus can be used for the characterization of the latter.

Instead of existence theorems, we now give some examples. As already mentioned, the one-dimensional harmonic oscillator spectrum (which is completely uniform, hence has no fluctuations at all) leads to Riemann's zeta function, where this continuation process indeed extends the function for any s. What happens, however, if fluctuations are present?

Consider the spectrum of the Laplacian on the sphere, which leads to the spectrum $l(l+1)$ with the multiplicity $2l+1$ for $l=0, 1, 2, 3,\ldots$. In this case the *original* series $Z(s)$ is given by

$$Z_1(s)=\underbrace{\left[\frac{1}{2^s}+\cdots+\frac{1}{2^s}\right]}_{3\text{ times}}+\underbrace{\left[\frac{1}{6^s}+\cdots+\frac{1}{6^s}\right]}_{5\text{ times}}+\cdots$$
$$+\underbrace{\left[\frac{1}{[l(l+1)]^s}+\cdots+\frac{1}{[l(l+1)]^s}\right]}_{(2l+1)\text{ times}}+\cdots$$

It is important to observe that (a) the zero eigenvalue is omitted (to avoid an obvious divergence), and (b) the terms are enumerated *explicitly* in order to apply the analytical extension scheme *to the original series at it stands* and not to any particular rearrangement of it (since such a rearrangement scheme is not available in general). In fact we must stress that the inclusion of a particular rearrangement leads in principle to a *new* analytical extension scheme, and may indeed be needed to generate an analytical extension or improve an already existing one. For example, in the present case we may replace each set of identical terms with *one* term multiplied by a degeneracy factor, such as

$$Z_2(s)=\sum_{l=1}^{\infty}(2l+1)/[l(l+1)]^s$$

In this way we have telescoped many terms into one, and reduced thereby the fluctuations. In fact, the general term of $Z_2(s)$ for large values of l behaves like the series for $2\zeta(2s-1)$; hence, the Euler–Maclaurin extension scheme is the best possible one for this new starting series $Z_2(s)$ [since it is the best one for $\zeta(s)$]. The difference between $Z_1(s)$ and $Z_2(s)$ resides in the large fluctuations engendered by enumerating in Z_1 the identical terms one by one. The staircase function $N(\mu)$, which counts the number of eigenvalues having value less than μ, jumps by $2l+1$ whenever μ is $l(l+1)$. These jumps measure the size of the fluctuations around the smooth curve that connects the midpoints of the steps, and they increase as the square root of the eigenvalues. Consequently, in the Euler–Maclaurin continuation [Eq. (21)] applied to the series Z_1, the partial sum $\sum_1^N \mu_n^{-s}$ will fluctuate as $\mu_N^{1/2}/\mu_N^s$, and we expect the limit $N\to\infty$ to exist only for $s>1/2$. Thus, the method still provides an extension from $s>1$, but only for an $s>1/2$. An extreme case is furnished by the sequence $\mu_n=2, 4, 4, 8, 8, 8, 8,\ldots$ (i.e., the set 2^l, with each term repeated 2^{l-1} times). Here, the fluctuations are of the order μ_n, and the Euler–Maclaurin method applied to this series provides no analytical extension at all for $s \leqslant 1$. (The latter example is only used to show the possible failure of the extension scheme; we know of no Hamiltonian with this sequence of eigenvalues).

We now apply these observations to *multidimensional, separable* problems. Here the *original* definition of the zeta function uses an enumeration of the eigenvalues according to increasing order, which implies large spectral fluctuations of the order $\mu_n^{1/2}$.[11] On the other hand, it is also clear that the *natural* enumeration should proceed according to the quantum numbers generated by the Bohr–Sommerfeld quantization rules applied to each degree of freedom. This results in a particular *rearrangement* of the original series, which suppresses these fluctuations.

6. PSEUDOSPHERICAL BILLIARDS

It has been conjectured that the nature of spectral fluctuations in a quantum system may reflect the chaotic nature of the corresponding classical motion.[5] The evidence for this conjecture rests entirely on numerical studies, which have mostly been carried out on plane Euclidean billiards. (In fact, we know of no simple model in which proven chaoticity is generated by nonsingular external forces.) A Euclidean billiard is a bounded plane domain in which the classical motion is the free motion of a point mass with specular reflections at the boundary, while the quantum Hamiltonian is defined as the negative of the Laplacian inside the domain, with Dirichlet conditions (for instance) on the boundary. The computational advantage of these models resides in the fact that the motion is free between reflections; this means, however, that all interesting dynamical effects arise from the *boundary* terms of the Hamiltonian, which are *singular*. This persists in the quantum theory as well, where the discrete spectrum is entirely generated by the boundary conditions.

Depending on the shape of the domain, the classical motion in a billiard can range from separable (e.g., the circular billiard) to chaotic (e.g., the stadium).[4,5] It has then been found numerically that the quantal spectra of those billiards that are classically chaotic exhibit different features from the others. In particular, the statistics of nearest neighbor spacings is Poissonian if the system is separable, implying the predominance of small separations. In examples that are classically chaotic, this preference for small separations is replaced by a preference for large separations, often referred to as "level repulsion"; in fact, these statistics seem to follow the ones obtained in random matrix theories.[5] The main qualitative effect of this property appears to be a reduction in the fluctuations of the levels around their mean distribution. As yet, it is entirely unclear from the theoretical point of view why this should be the case.

In dynamical systems, the features responsible for the generation of chaoticity do not reside entirely in the boundary. We turn now to models in which the causes of classical chaoticity are more evenly distributed. The free, or geodesic, motion on a surface of *negative curvature* is *exponentially separating*, thus containing *one of the basic ingredients* of chaoticity. The quantum Hamiltonian is simply the negative of the Laplace–Beltrami operator. *The other major ingredient for chaoticity* is the *compactness of the phase space*; this we can achieve either by generalized periodicity conditions or by cutting out a portion of the surface and using it as a billiard, attaching suitable boundary conditions to the edges. In the simplest case, we take the negative curvature *constant* (equal to -1 if we choose the length unit appropriately), introducing thereby *pseudospherical* billiards.[6] (These can be pictured as portions of the familiar Poincaré upper half-plane or of the Poincaré disk, where the geodesics are arcs of circles or straight line segments.)

However, we must stress that far from all pseudospherical billiards are chaotic; for example, convex edges may refocus the exponentially divergent trajectories; this is immediately visible for a circular billiard, which is separable in the Poincaré disk variables and hence is certainly not chaotic.

The pseudospherical billiards with geodesic edges (the *polygonal* billiards) may be chaotic. In addition, these billiards may or may not tessellate the full Poincaré surface under repeated reflections across the edges (here reflections are realized as inversions around the geodesic edges). The *tessellating polygonal billiards* have desirable features: (a) they are certainly chaotic classically, (b) their quantal spectrum is in a deep relationship with the classical periodic orbits, exhibited by *Selberg's trace formula*. The latter in particular provides us at once with the totality of the expansion coefficients c_δ needed to describe the analytical structure of $Z(s)$.[6] In the Selberg trace formula, however, it is not the Laplace–Beltrami operator that appears naturally, but the combination $(-\Delta - 1/4)$; for this reason it is convenient to choose our sequence $\{\mu_n\}$ as the eigenvalues of *this* operator, and we shall do that; for a singly connected billiard with Dirichlet boundary conditions these eigenvalues are *all positive*.[12] The Selberg trace formula for a tessellating billiard with the same boundary conditions then yields the following expression for the partition function:

$$\Theta(t) = \sum_n e^{-t\mu_n} = \operatorname{Tr} \exp\left[\left(\Delta + \frac{1}{4}\right) t\right]$$
$$\sim \frac{c_{-1}}{t} + \frac{c_{-1/2}}{\sqrt{t}} + \sum_{n=0}^{\infty} c_n t^n \tag{24}$$

with

$$c_{-1} = \frac{\text{Area}}{4\pi} \tag{25a}$$

$$c_{-1/2} = \frac{-\text{Circumference}}{8\sqrt{\pi}} \tag{25b}$$

$$c_0 = -\frac{\text{Area}}{48\pi} + \frac{1}{24}\sum_{\{r\}}\left(m_r - \frac{1}{m_r}\right) \tag{25c}$$

and in general

$$c_n = -\frac{\text{Area}}{4\pi}(1 - 2^{-1-2n})\frac{B_{2n+2}}{(n+1)!}$$
$$-\frac{(2n)!}{n!}\sum_{\substack{p+q=n\\ p,q\geqslant 0}}\frac{B_{2p}B_{2q+2}}{(2p)!\,(2q+2)!}\left(\frac{1-2^{1-2p}}{2}\right)\sum_{\{r\}}\frac{m_r^{2q+2}-1}{m_r} \tag{26}$$

for $n = 0, 1, 2,...,$ where $\sum_{\{r\}}$ is a sum over the corners, with π/m_r being the rth corner angle; the tessellation implies that m_r is an integer.

For nontessellating billiards the situation is much less favorable. There, only the coefficients c_{-1}, $c_{-1/2}$, and c_0 can be simply determined by using the universal asymptotic distribution of eigenvalues given by Weyl[1] (see Appendix); moreover, positive half-integral powers of t can also appear.

We may now apply the general considerations of the previous sections to these billiards, since the general conditions implied for their validity are met. Since the leading term in $\Theta(t)$ is proportional to $1/t$ for any billiard, $\alpha = -1$; hence, the defining series for $Z(s)$ always converges for $\operatorname{Re} s > 1$. For $s \leqslant 1$, a continuation process is needed, and we have yet *no theoretical clue* how the spectral fluctuations actually affect it. In the following section we will then study the zeta function $Z(s)$ *numerically* on selected examples of polygonal pseudospherical billiards.

(It is also of intrinsic interest to study the same questions upon the spectra associated with compact surfaces of constant negative curvature, where the Selberg trace formula is equally helpful and even simpler; but for these particular spectra, the numerical results are not yet sufficiently complete.)

7. NUMERICAL EVIDENCE ON PSEUDOSPHERICAL BILLIARDS

The billiard used in these calculations is a triangular billiard cut out of a surface of constant negative curvature -1, with corner angles $\pi/2$, $\pi/3$, $\pi/8$, circumference

$$\operatorname{arsinh} 2^{-1/4} + \operatorname{artanh} 2^{-1/4} \simeq 1.9885117$$

and area $\pi/24$. (This billiard tessellates a compact surface of constant negative curvature of genus two, and thereby the classical motion in it is chaotic.[6])

As explained in the previous section, the numbers μ_n are the eigenvalues of $(-\Delta - 1/4)$ in this triangular domain with Dirichlet boundary conditions.

The numerical study is based on a table of about 1500 consecutive eigenvalues computed by Schmit.[13] For orientation, $\mu_0 \simeq 223.0$ and $\mu_{1548} \simeq 154452.9$ (the decimal digit is uncertain throughout the table). The agreement with the Weyl distribution has already been used to check that the precise number of levels has been found.

The defining series $\sum \mu_n^{-s}$ for $Z(s)$, $s > 1$, converges very rapidly.

For the analytical extension we use Eq. (21), which results in the following expression:

$$Z(s) = \lim_{N \to \infty} \left[\sum_{n=0}^{N-1} \mu_n^{-s} - C_M(N, s) \right] \tag{27}$$

with

$$C_M(N, s) = -\frac{1}{\mu_N^s} \left[\frac{c_{-1}}{s-1} \frac{\mu_N}{\Gamma(1)} + \frac{c_{-1/2}}{s-1/2} \frac{\mu_N^{1/2}}{\Gamma(1/2)} + \frac{1}{2} \right] + \sum_{m=1}^{(M+1)/2} \frac{B_{2m}}{(2m)!} \left(\frac{d}{dW} \right)^{2m-1} \left(\frac{1}{\mu^s} \right) \Bigg|_{\mu = \mu_N} \tag{28}$$

where $W(\mu)$ is the explicit form of the Weyl expression (18),

$$W(\mu) = \frac{c_{-1}}{\Gamma(2)} \mu + \frac{c_{-1/2}}{\Gamma(3/2)} \mu^{1/2} + c_0 \tag{29}$$

and the coefficients c_α are computed from Eq. (25); for this particular billiard,

$$c_{-1} = 1/96, \quad c_{-1/2} \simeq 0.1402372, \quad c_0 = 577/1152 \tag{30}$$

Thus, at $s = 1$, $Z(s)$ has a simple pole of residue 1/96. The finite part of

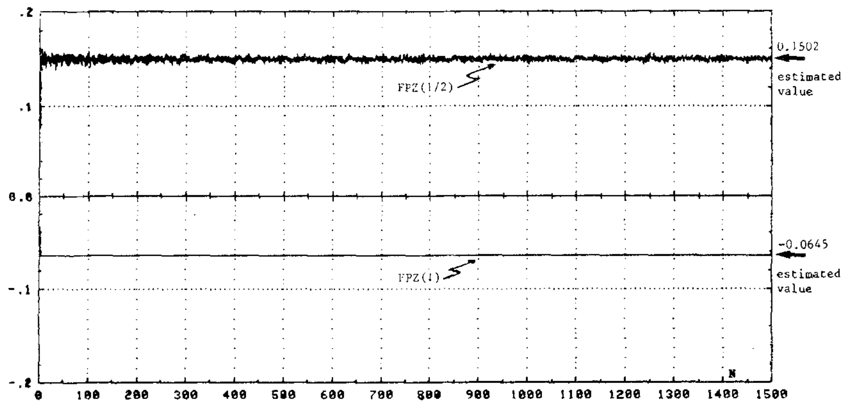


Fig. 1. Numerical estimates of the finite parts of Z(1) and Z(1/2) as a function of the number of terms in the partial sums.

$Z(s)$ at $s=1$, FPZ(1), is defined as the limit of $Z(s)-(1/96)(s-1)^{-1}$. From Eqs. (27) and (28) we obtain

$$\mathrm{FPZ}(1)=\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{1}{\mu_n}-c_{-1}\log\mu_n\right) \tag{31}$$

[this obviously gives a generalization of Euler's constant as defined by Eq. (5)].

The numerical computation still converges rapidly to the value -0.064509. In fact, only about 20 terms suffice to achieve an accuracy of

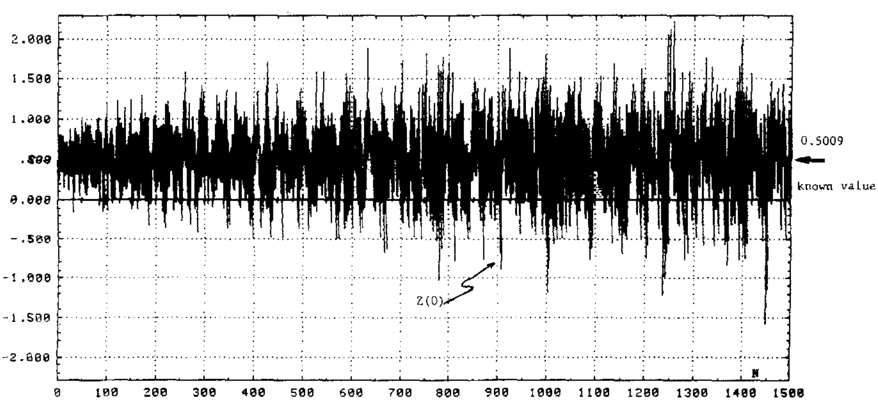


Fig. 2. Numerical estimates of Z(0) as a function of the number of terms in the partial sums. This curve also describes the fluctuation at the Nth level in the spectral staircase around the Weyl distribution without the constant term; see Eq. (23).

99% (see Fig. 1). The same considerations can be applied at $s = 1/2$ (the location of the next pole), giving

$$\mathrm{FPZ}\left(\frac{1}{2}\right) = \lim_{N \to \infty} \left[\sum_{n=0}^{N-1} \mu_n^{-1/2} - 2c_{-1}\mu_N^{1/2} - \frac{c_{-1/2}}{\Gamma(1/2)} \log \mu_N \right] \tag{32}$$

The numerical evaluation results in 0.15019. Figure 1 shows, however, that the rate of convergence is much diminished as the fluctuations are now being amplified. In fact, to reach the same accuracy of 99%, about 70 terms are now needed.

As s further decreases, the fluctuations are increasingly amplified and for this spectrum convergence ceases at the value $s = 0$ (Fig. 2), where the trace identity (14) gives the value $Z(0) = c_0 = 577/1152$.

A glance at Fig. 2 shows that these fluctuations are moderate, and steady around the correct limiting value. According to Eq. (23), the same fluctuations at $s = 0$ give the difference between the Weyl formula and the actual eigenvalue distribution. Thus, on the one hand these deviations are *large* enough to destroy the convergence of the present extension scheme, while on the other hand they are *small* enough to suggest an improved method using an additional averaging to obliterate the steady fluctuations. A simple local Gaussian averaging can indeed be used (involving around 50 levels) to find the value 0.5008(2) as an approximation to the correct value $577/1152 = 0.500868...$ (Fig. 3). In the expression of the spectral staircase as a sum of oscillating terms (the periodic orbit sum, see Berry[4]), the averaging enhances the slowest oscillations, which thereby become visible on the figures.

{In quantum field theories one also encounters the functional determinant,[14] i.e., the renormalized value of the expression $\prod_{n=0}^{\infty} \mu_n$, which is then defined as the limit of $\exp[-Z'(s)]$ as $s \to 0$. In the present problem the same method gives

$$Z'(0) = \lim_{N \to \infty} \left[\sum_{n=0}^{N-1} (-\log \mu_n) + c_{-1}\mu_N(\log \mu_N - 1) + \frac{c_{-1/2}\mu_N^{1/2}}{\Gamma(1/2)} (2 \log \mu_N - 4) - \frac{1}{2} \log \mu_N \right] \tag{33}$$

which is a generalization of the Stirling formula implied in Eq. (6). A Gaussian smoothing then gives the numerical value $Z'(0) = -0.22(5)$; see Fig. 4. We stress that while an analytical method exists to find $Z(0)$, no such method is known to us to find $Z'(0)$.}

As we decrease s below zero, the extension scheme with Gaussian smoothing becomes increasingly more unstable and more sensitive to the numerical uncertainties in the μ_n sequence. The next place where the accuracy can be checked against the value obtained through a trace identity is at $s = -1$, where

$$Z(-1) = 110599/138240 \simeq 0.80005$$

However, in view of Fig. 5, we have reached the limit of our capabilities, and the method proposed requires a substantial improvement, both in the numerical accuracy of the data and in the number of eigenvalues con-

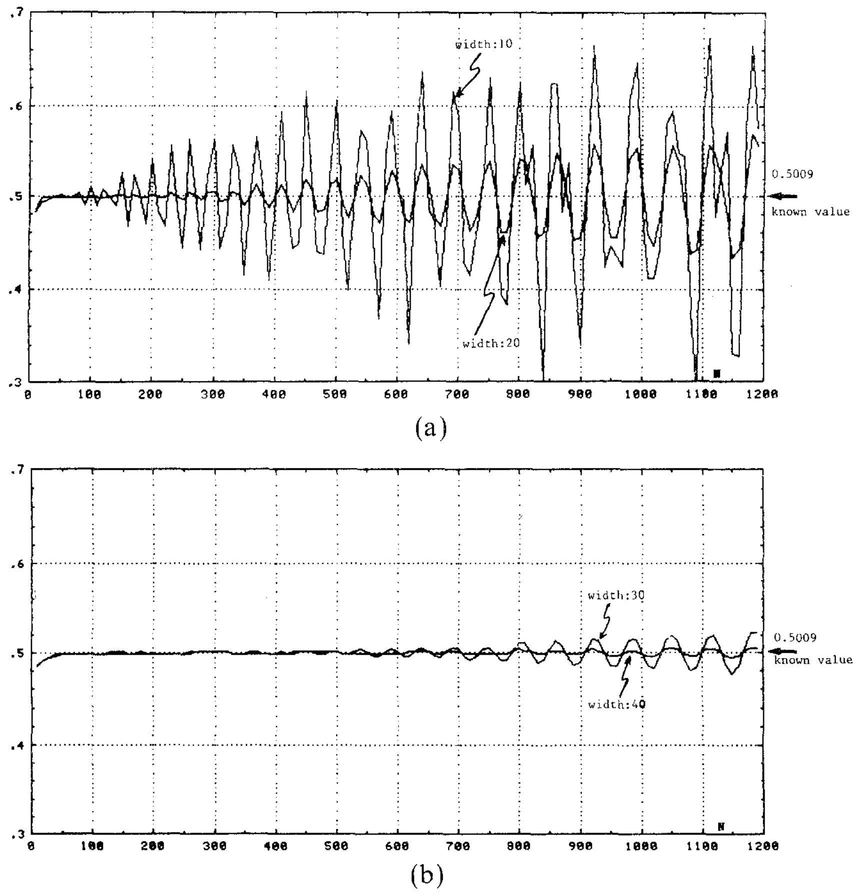
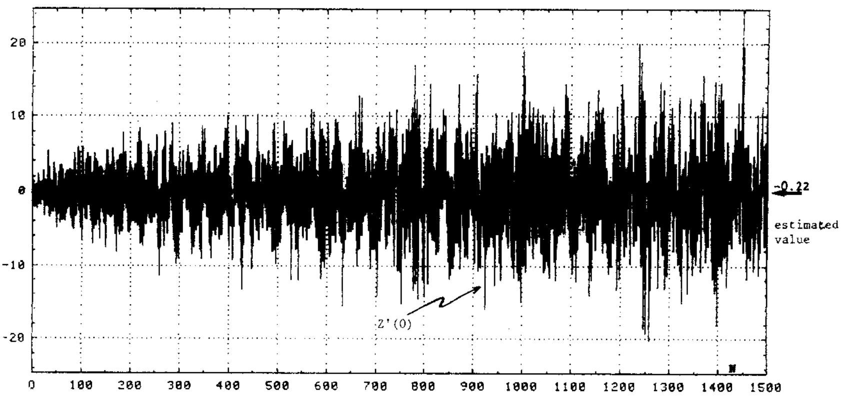


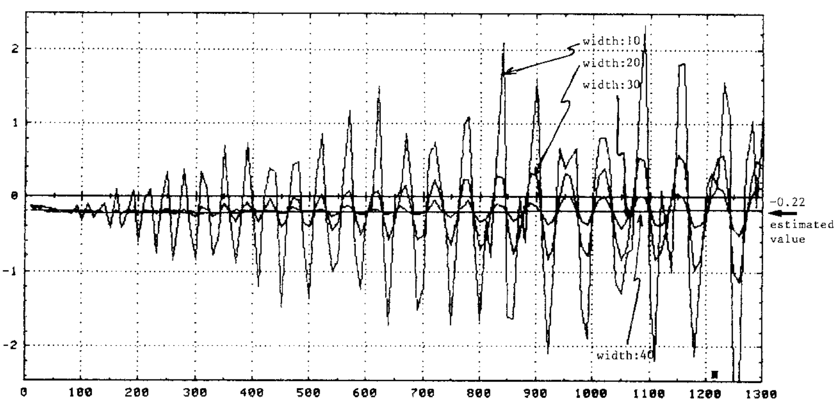
Fig. 3. The curve in Fig. 2 smoothed through a convolution with Gaussians of different widths; (a) 10 and 20, (b) 30 and 40. The remaining oscillations reflect the contribution of the shortest periodic geodesics to the spectral staircase. Note the change in the vertical scale by a factor ten compared to Fig. 2.

sidered. The preliminary estimate from the figure suggests a value $-2 \leqslant Z(-1) \leqslant +1$. This estimate is found by studying the first 300 eigenvalues, where it comes about as the difference of two large numbers of the order of 10^7. One may well wonder how to reconcile this accuracy with the precision with which the individual levels are known. In fact, taking into account more eigenvalues makes the error larger, and we must resort to an averaging procedure to increase the accuracy.

If no averaging is done, the error obtained in a partial sum of length N is of order $N\varepsilon_N$, ε_N being the error on the Nth level (here less then one unit). If we average the partial sum over $2K$ levels around N, this error is



(a)



(b)

Fig. 4. (a) Numerical estimates of Z′(0) as a function of the number of terms in the partial sums. (b) The same curve smoothed through a convolution with Gaussians of width 10, 20, 30, and 100. Note the change in the vertical scale.

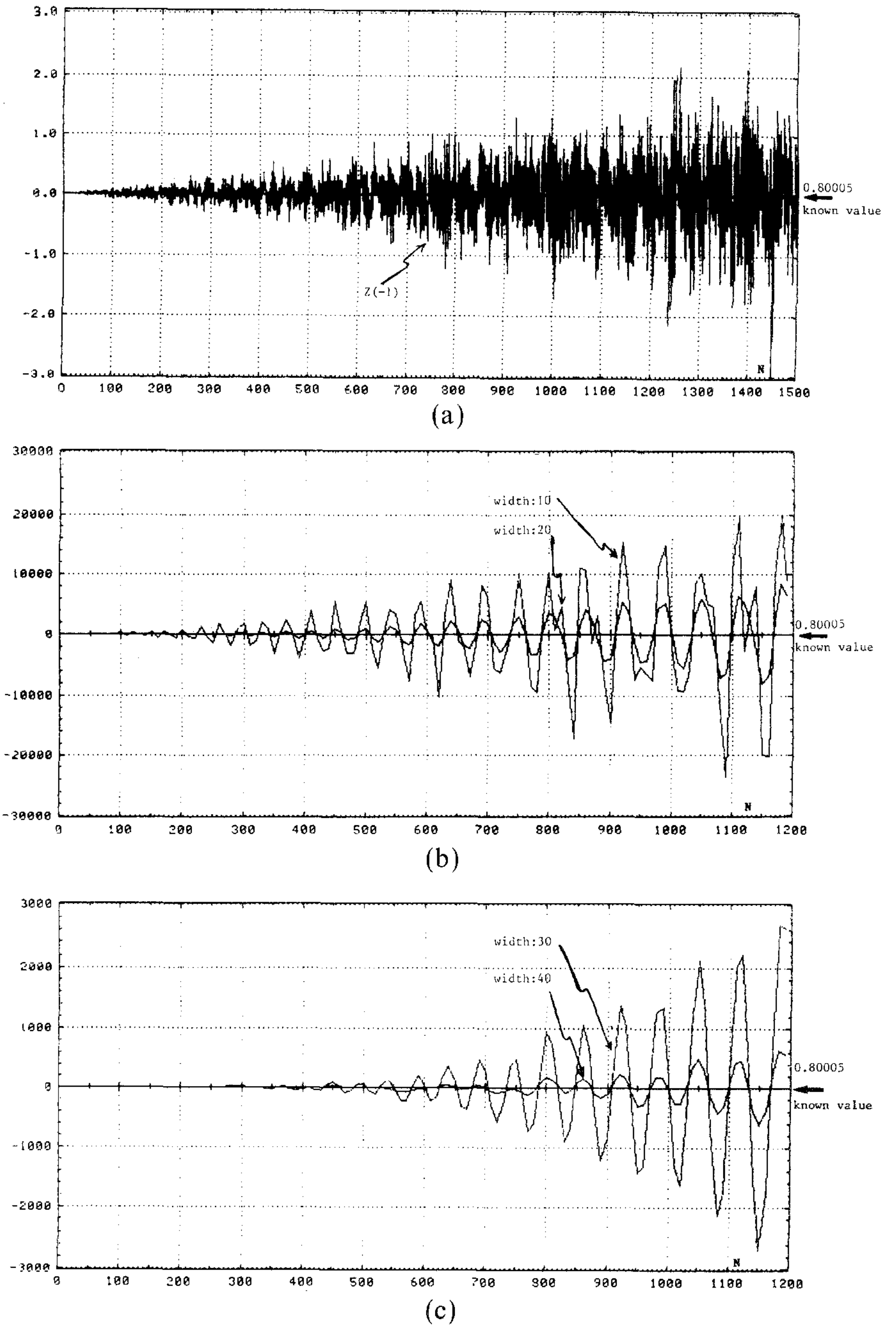
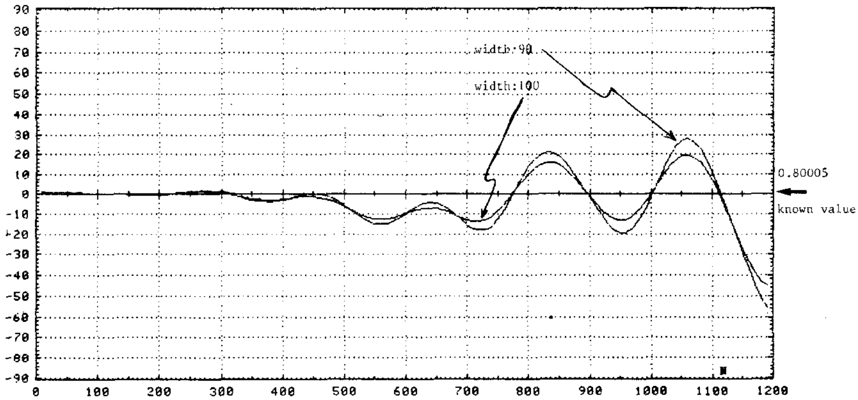
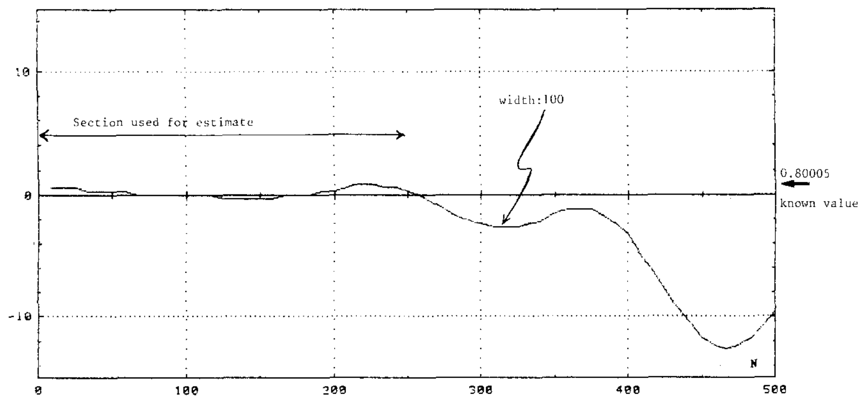


Fig. 5. (a) Numerical estimates of $Z(-1)$ as a function of the number of terms in the partial sums. The vertical scale is in units of 10^5! Also shown is the same curve smoothed through a convolution with Gaussians of different widths, plotted on different scales: (b) widths 10, 20: scale 0 to 3×10^4; (c) widths 30, 40: scale 0 to 3×10^3; (d) widths 90, 100: scale 0 to 90. (e) The suggested value between -2 and 1 obtained from an enlargement of part (d) (scale 0 to 15).



(d)



(e)

Fig. 5 (*continued*)

reduced to $N[\langle \varepsilon^2 \rangle / 2K]$. A more detailed statistical analysis of the data indicates that $\langle \varepsilon^2 \rangle$ is small enough to give a sensible convergence to the partial sum with an error of 50% in the result. This analysis is further supported by the study of the fluctuations in the partial sums at $s = 0$, and using these estimates to evaluate $\langle \varepsilon^2 \rangle$ used at $s = -1$. However, with the present numerical accuracy, we can proceed no further with the analytical continuation than $s = -1$.

It is interesting to compare this situation with the analytical continuation of Riemann's zeta function. There the "spectrum" is completely regular, being simply the succession of positive integers, and the complete lack of fluctuations allows one to extend the Euler–Maclaurin extension scheme to smaller values of s (typically $s \simeq -10$).

8. SUMMARY

We studied the convergence of an analytical continuation process applied to a zeta function $Z(s)$ to characterize the spectral fluctuations and found that this process is made considerably more unstable by the presence of fluctuations. For a regular spectrum previous results indicated that value of s between -5 and -10 are easily accessible. For a (chaotic) pseudospherical billiard and with similar numerical accuracy we already found $s \sim -1$ hard to reach. We also gave arguments indicating that for the spectra of integrable systems (with Poissonian nearest neighbor gap fluctuations) the analytical process is even more unstable.

APPENDIX

We explain the algebraic connections between the large-μ expansion of the smoothed eigenvalue counting function $W(\mu)$ and the small-t expansion of the partition function $\Theta(t)$.

For t small, $\Theta(t)$ has the expansion given by the integral

$$\Theta(t) \sim \int_0^\infty e^{-t\mu}\, dW(\mu) = \int_0^\infty e^{-z}\, dW(z/t) \tag{A.1}$$

Hence, for small t, only the asymptotic behavior of $W(\mu)$ for large μ will be of importance. Let this be given as

$$W(\mu) \sim w_{\alpha'} \mu^{-\alpha'} + \omega_{\beta'} \mu^{-\beta'} + \cdots \tag{A.2}$$

Inserting this into (A.1), we immediately obtain, for small t,

$$\Theta(t) \sim w_{\alpha'} \Gamma(1-\alpha')\, t^{\alpha'} + w_{\beta'} \Gamma(1-\beta')\, t^{\beta'} + \cdots \tag{A.3}$$

By comparison with Eq. (10), i.e.,

$$\Theta(t) \sim c_\alpha t^\alpha + c_\beta t^\beta + \cdots$$

we find that $\alpha' = \alpha$, $\beta' = \beta$, etc., and

$$w_\alpha = c_\alpha / \Gamma(1-\alpha), \qquad w_\beta = c_\beta / \Gamma(1-\beta), \qquad \text{etc.} \tag{A.4}$$

Let the eigenvalue sequence $\{\lambda_n\}$ refer to the operator $(-\Delta)$, where Δ is the Laplace (or Laplace–Beltrami) operator with Dirichlet boundary conditions on a bounded two-dimensional domain. Then $W(\lambda)$ is the asymptotic Weyl distribution; its first three terms have a simple and universal geometrical meaning, giving

$$c_{-1}=\frac{\text{Area}}{4\pi} \tag{A.5}$$

$$c_{-1/2}=-\frac{\text{Circumference}}{8\sqrt{\pi}} \tag{A.6}$$

$$\begin{aligned} c_0= & +\frac{1}{12\pi}\iint K\,d^2\sigma \quad \text{(surface Gaussian curvature term)} \\ & -\frac{1}{24\pi}\int J\,ds \quad \text{(boundary mean curvature term)} \\ & +\frac{1}{24}\sum_{\{r\}}\left(\frac{\pi}{\alpha_r}-\frac{\alpha_r}{\pi}\right) \quad \text{(corner angle term)} \end{aligned} \tag{A.7}$$

(the precise definitions are given in Ref. 15, where it is shown that c_0 implicitly accounts for the *connectivity* of the domain as well).

In the case of a polygonal billiard on a surface of constant negative curvature -1, it is more convenient to study the spectrum $\{\mu_n=\lambda_n-1/4\}$ of the operator $(-\Delta-1/4)$. This shift of variable alters the coefficient c_0 to

$$c_0=-\frac{\text{Area}}{48\pi}+\frac{1}{24}\sum_r\left(m_r-\frac{1}{m_r}\right) \tag{A.8}$$

where we have also simplified Eq. (A.7) using $K\equiv -1$ and $J\equiv 0$, and set $m_r=\pi/\alpha_r$ (an integer if the billiard generates a tessellation).

ACKNOWLEDGMENTS

N.L.B. thanks the National Science Foundation, the Commissariat de l'Energie Atomique, and the Ministère des Relations Extérieures for partial financial support. He is also grateful to the Service de Physique Théorique of Saclay for their extensive and kind hospitality. All three authors appreciate the kind invitation to celebrate Mark Kac's memory.

REFERENCES

1. H. P. Baltes and E. R. Hilf, *Spectra of Finite Systems* (Bibliographisches Institut, Mannheim, 1976).
2. M. Kac, *Am. Math. Month.* **73S**:1 (1966).
3. V. I. Arnold and A. Avez, *Ergodic Problems in Classical Mechanics* (Benjamin, 1968), Chapter 3.

4. M. V. Berry, in *Chaotic Behaviour of Deterministic Systems* (Proceedings XXXVI. Les Houches 1981), G. Iooss *et al.*, eds. (North-Holland, 1983), p. 173; in *Chaotic Behaviour in Quantum Systems* (Proceedings, Como 1983), G. Casati, ed. (Plenum Press, New York, 1985).
5. O. Bohigas, M. J. Giannoni, and C. Schmit, *Phys. Rev. Lett.* **52**:1 (1984); O. Bohigas and M. J. Giannoni, in *Mathematical and Computational Methods in Nuclear Physics*, J. S. Dehesa *et al.*, eds. (Springer, 1984).
6. N. L. Balazs and A. Voros, *Phys. Rep.* **143**(3):109 (1986).
7. E. C. Titchmarsh, *The Theory of the Riemann Zeta Function* (Oxford University Press, 1951).
8. H. M. Edwards, *Riemann's Zeta Function* (Academic Press, 1974).
9. L. A. Dikii, *Usp. Mat. Nauk* **13**:111 (1958) (Translation AMS Series 2, Vol. 18, p. 81).
10. A. Voros, *Nucl. Phys. B* **165**:209 (1980).
11. M. V. Berry and M. Tabor, *Proc. R. Soc. A* **356**:375 (1977).
12. H. P. McKean, *J. Diff. Geom.* **4**:359 (1970).
13. C. Schmit, to be published.
14. S. W. Hawking, *Commun. Math. Phys.* **56**:133 (1977); E. D'Hoker and D. H. Phong, *Commun. Math. Phys.* **104**:537 (1986).
15. H. P. McKean and I. M. Singer, *J. Diff. Geom.* **1**:43 (1967).